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Equality solve mismatch and order details, for

RESIDUAL-DRIVEN ADAPTIVE MULTI-RATE QUADRATIC PROGRAMMING FRAMEWORK FOR NONLINEAR ANALOG AUDIO CIRCUIT EMULATION

Starting from eq. (20) in the paper, we re-state the equality solve mismatch

$$\mathbf{A}_{\text{eq}}^{(k)}\Delta\mathbf{z}_k = \mathbf{e}_k, \qquad \Delta\mathbf{z}_k = \mathbf{z}_k - \mathbf{z}_{k-1}.$$

where the equality matrix and the corresponding vector are

$$\mathbf{A}_{\text{eq}}^{(k)} = \begin{bmatrix} \mathbf{I} - h_k\mathbf{A} & -h_k\mathbf{G} \\ -\mathbf{D} & \mathbf{H} \\ \mathbf{0} & \mathbf{J}_{k-1} \end{bmatrix}, \qquad \mathbf{b}_{\text{eq}}^{(k)} = \begin{bmatrix} \mathbf{x}_{k-1} + h_k\mathbf{B}u_k \\ \mathbf{E}u_k \\ h_k\mathbf{K}_k\boldsymbol{\eta}(\mathbf{q}_{k-1}) + \mathbf{J}_{k-1}\mathbf{q}_{k-1} \end{bmatrix}.$$

Therefore,

$$\mathbf{A}_{\text{eq}}^{(k)}\Delta\mathbf{z}_k = \mathbf{A}_{\text{eq}}^{(k)}\mathbf{z}_k - \mathbf{A}_{\text{eq}}^{(k)}\mathbf{z}_{k-1} = \mathbf{b}_{\text{eq}}^{(k)} - \mathbf{A}_{\text{eq}}^{(k)}\mathbf{z}_{k-1}.$$

Now,

$$\mathbf{A}_{\text{eq}}^{(k)}\mathbf{z}_{k-1} = \begin{bmatrix} \mathbf{I} - h_k\mathbf{A} & -h_k\mathbf{G} \\ -\mathbf{D} & \mathbf{H} \\ \mathbf{0} & \mathbf{J}_{k-1} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{k-1} \\ \mathbf{q}_{k-1} \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{k-1} - h_k\mathbf{A}\mathbf{x}_{k-1} - h_k\mathbf{G}\mathbf{q}_{k-1} \\ -\mathbf{D}\mathbf{x}_{k-1} + \mathbf{H}\mathbf{q}_{k-1} \\ \mathbf{J}_{k-1}\mathbf{q}_{k-1} \end{bmatrix}.$$

Hence,

$$\mathbf{e}_k = \mathbf{b}_{\text{eq}}^{(k)} - \mathbf{A}_{\text{eq}}^{(k)}\mathbf{z}_{k-1} = \begin{bmatrix} \cancel{\mathbf{x}_{k-1}} + h_k\mathbf{B}\mathbf{u}_k - \cancel{\mathbf{x}_{k-1}} + h_k\mathbf{A}\mathbf{x}_{k-1} + h_k\mathbf{G}\mathbf{q}_{k-1} \\ \mathbf{E}\mathbf{u}_k + \mathbf{D}\mathbf{x}_{k-1} - \mathbf{H}\mathbf{q}_{k-1} \\ h_k\mathbf{K}_k\boldsymbol{\eta}(\mathbf{q}_{k-1}) + \cancel{\mathbf{J}_{k-1}\mathbf{q}_{k-1}} - \cancel{\mathbf{J}_{k-1}\mathbf{q}_{k-1}} \end{bmatrix}.$$

Using the constraint at the previous step (eq. (10b) in the paper),

$$-\mathbf{D}\mathbf{x}_{k-1} + \mathbf{H}\mathbf{q}_{k-1} = \mathbf{E}\mathbf{u}_{k-1},$$

and taking eq. (18)

$$\mathbf{K}_k = -\frac{\mathbf{I}}{h_k},$$

we obtain

$$\mathbf{e}_k = \begin{bmatrix} h_k\left(\mathbf{A}\mathbf{x}_{k-1} + \mathbf{B}\mathbf{u}_k + \mathbf{G}\mathbf{q}_{k-1}\right) \\ \mathbf{E}\left(\mathbf{u}_k - \mathbf{u}_{k-1}\right) \\ -\boldsymbol{\eta}(\mathbf{q}_{k-1}) \end{bmatrix} := \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}.$$

Bounds on the three mismatch terms

For the first term,

$$\|\mathbf{e}_1\| = \|h_k\left(\mathbf{A}\mathbf{x}_{k-1} + \mathbf{B}\mathbf{u}_k + \mathbf{G}\mathbf{q}_{k-1}\right)\| = h_k\left\|\mathbf{A}\mathbf{x}_{k-1} + \mathbf{B}\mathbf{u}_k + \mathbf{G}\mathbf{q}_{k-1}\right\|,$$

$$\|\mathbf{e}_1\| \leq h_k\left(\|\mathbf{A}\mathbf{x}_{k-1}\| + \|\mathbf{B}\mathbf{u}_k\| + \|\mathbf{G}\mathbf{q}_{k-1}\|\right) = h_k\left(\|\mathbf{A}\|\|\mathbf{x}_{k-1}\| + \|\mathbf{B}\|\|\mathbf{u}_k\| + \|\mathbf{G}\|\|\mathbf{q}_{k-1}\|\right),$$

then, from the bounds on the signals (Section 3.1 in the paper)

$$\Rightarrow \|\mathbf{e}_1\| \leq h_k\left(\|\mathbf{A}\|X_{\max} + \|\mathbf{B}\|U_{\max} + \|\mathbf{G}\|Q_{\max}\right).$$

Thus,

$$\|\mathbf{e}_1\| \sim \mathcal{O}(h_k).$$

For the second term, using again the bounds on the signals established in Section 3.1 of the paper

$$\|\mathbf{E}(\mathbf{u}_k - \mathbf{u}_{k-1})\| \leq \|\mathbf{E}\|\,\|\mathbf{u}_k - \mathbf{u}_{k-1}\| \leq \|\mathbf{E}\|U_{BL}h_k.$$

Therefore,

$$\|\mathbf{e}_2\| \sim \mathcal{O}(h_k).$$

For the third term,

$$\|-\boldsymbol{\eta}(\mathbf{q}_{k-1})\| = \|\boldsymbol{\eta}(\mathbf{q}_{k-1})\| \sim \mathcal{O}(h_{k-1}^2),$$

by induction, i.e., if we assume the proposition that the residual scales quadratically with respect to the step-size to be true, then the previous statement holds, inductively.
Moreover, if (from eq. (17) in the paper)

$$\frac{h_k}{h_{k-1}} \geq \frac{1}{\gamma_h},$$

then

$$\gamma_h h_k \geq h_{k-1}.$$

Hence,

$$\|\mathbf{e}_3\| \sim \mathcal{O}(h_{k-1}^2) \quad \Rightarrow \quad \|\mathbf{e}_3\| \leq C h_{k-1}^2 \leq C\gamma_h^2 h_k^2,$$

and therefore

$$\|\mathbf{e}_3\| \sim \mathcal{O}(h_k^2).$$

Finally,

$$\|\mathbf{e}_k\| \leq \|\mathbf{e}_1\| + \|\mathbf{e}_2\| + \|\mathbf{e}_3\| \leq \mathcal{O}(h_k) + \mathcal{O}(h_k) + \mathcal{O}(h_k^2).$$

Hence, the order of $\|\mathbf{e}_k\|$ will be the max order of the three terms which, given $0 < h_k < 1$, is $\mathcal{O}(h_k)$.
Consequently,

$$\boxed{\|\mathbf{e}_k\| \sim \mathcal{O}(h_k).}$$