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Circuit matrix derivations for

RESIDUAL-DRIVEN ADAPTIVE MULTI-RATE QP EMULATION OF NONLINEAR ANALOG CIRCUITS FOR AUDIO

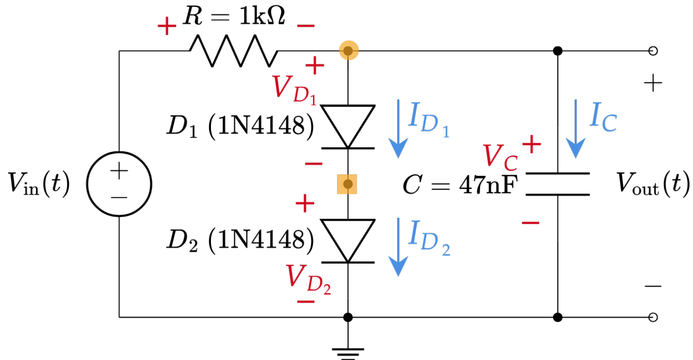


Figure 1: Diode clipper circuit.

Circuit equations using the quantities defined in Figure 1:

$$V_{D_1} + V_{D_2} - V_C = 0, \tag{1a}$$

$$\frac{V_{\text{in}} - V_C}{R} - I_{D_1} - I_C = 0, \tag{1b}$$

$$I_{D_1} - I_{D_2} = 0. \tag{1c}$$

Change of variables:

$$x = V_C, \quad u = V_{\text{in}}, \quad y = V_{\text{out}} = V_C, \quad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} V_{D_1} \\ V_{D_2} \\ I_{D_1} \\ I_{D_2} \end{bmatrix}. \tag{2}$$

Under (2), equations (1b), (1a) and (1c) respectively become:

$$\dot{x} = -\frac{1}{RC}x + \frac{1}{RC}u - \frac{1}{C}q_3, \tag{3a}$$

$$q_1 + q_2 = x, \tag{3b}$$

$$q_3 - q_4 = 0. \tag{3c}$$

From (3a)–(3c) and (2), the state, linear constraint and output matrices are extracted as:

$$\mathbf{A} = -\frac{1}{RC}, \qquad \mathbf{B} = \frac{1}{RC}, \qquad \mathbf{G} = \begin{bmatrix} 0 & 0 & -1/C & 0 \end{bmatrix}, \tag{4a}$$

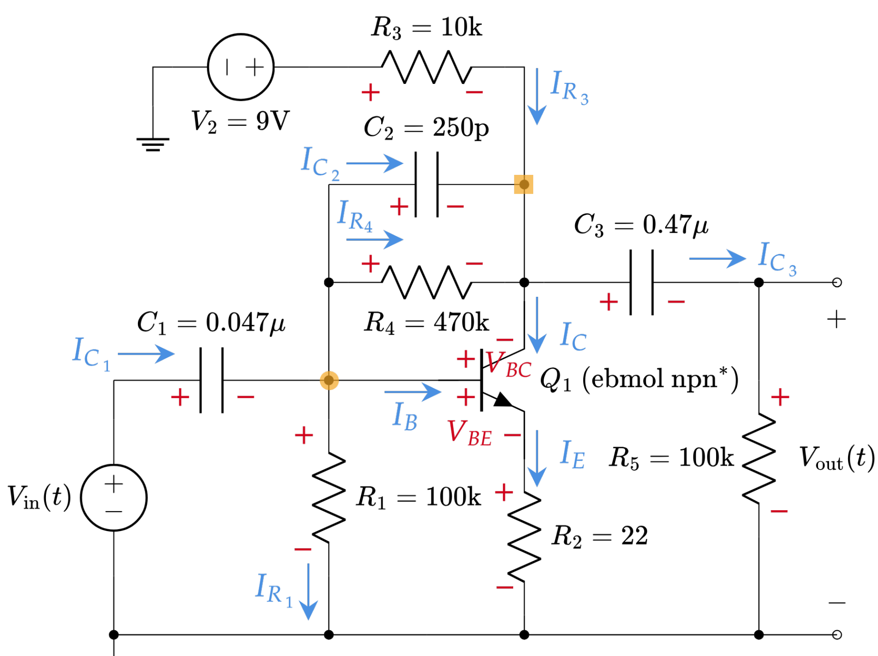
$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{E} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{4b}$$

$$\mathbf{M} = 1, \qquad \mathbf{L} = 0, \qquad \mathbf{N} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4c}$$

These, alongside the nonlinear constraints and Jacobian:

$$\boldsymbol{\eta}(\mathbf{q}) = \begin{bmatrix} q_3 - I_s\left(e^{q_1/(nV_T)} - 1\right) \\ q_4 - I_s\left(e^{q_2/(nV_T)} - 1\right) \end{bmatrix} = \mathbf{0}, \quad \frac{d\boldsymbol{\eta}(\mathbf{q})}{d\mathbf{q}} = \frac{I_s}{nV_T}\begin{bmatrix} -e^{q_1/(nV_T)} & 0 & \frac{nV_T}{I_s} & 0 \\ 0 & -e^{q_2/(nV_T)} & 0 & \frac{nV_T}{I_s} \end{bmatrix}, \tag{5}$$

define the complete model for the diode clipper circuit.



$^*(I_s = 6.734 \times 10^{-15},\ \beta_F = 200,\ \beta_R = 0.1)$

Figure 2: Common-emitter amplifier circuit.

Circuit equations using the quantities defined in Figure 2:

$$V_{\text{in}} - V_{C_1} - R_1 I_{R_1} = 0, \tag{6a}$$

$$R_1 I_{R_1} - V_{BE} - R_2 I_E = 0, \tag{6b}$$

$$V_{BC} - R_4 I_{R_4} = 0, \tag{6c}$$

$$V_{BC} - V_{C_2} = 0, \tag{6d}$$

$$V_2 - R_3 I_{R_3} - V_{C_3} - R_5 I_{C_3} = 0, \tag{6e}$$

$$I_{C_1} = I_{R_1} + I_B + I_{R_4} + I_{C_2}, \tag{6f}$$
$$I_{R_3} + I_{C_2} + I_{R_4} = I_C + I_{C_3}, \tag{6g}$$
$$I_B = I_E - I_C, \tag{6h}$$
$$V_2 - R_3 I_{R_3} + V_{C_2} - R_1 I_{R_1} = 0, \tag{6i}$$
$$V_{\text{out}} = R_5 I_{C_3}, \tag{6j}$$
$$I_{C_1} = C_1 \dot{V}_{C_1}, \tag{6k}$$
$$I_{C_2} = C_2 \dot{V}_{C_2}, \tag{6l}$$
$$I_{C_3} = C_3 \dot{V}_{C_3}. \tag{6m}$$

Change of variables:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} V_{C_1} \\ V_{C_2} \\ V_{C_3} \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} V_{\text{in}} \\ V_2 \end{bmatrix}, \qquad y = V_{\text{out}}, \qquad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} V_{BE} \\ V_{BC} \\ I_E \\ I_C \end{bmatrix}. \tag{7}$$

Under (7), equations (6a)–(6i) become:

$$u_1 - x_1 - R_1 I_{R_1} = 0, \tag{8a}$$
$$R_1 I_{R_1} - q_1 - R_2 q_3 = 0, \tag{8b}$$
$$q_2 - R_4 I_{R_4} = 0, \tag{8c}$$
$$q_2 - x_2 = 0, \tag{8d}$$
$$u_2 - R_3 I_{R_3} - x_3 - R_5 C_3 \dot{x}_3 = 0, \tag{8e}$$
$$C_1 \dot{x}_1 = I_{R_1} + q_3 - q_4 + I_{R_4} + C_2 \dot{x}_2, \tag{8f}$$
$$I_{R_3} + C_2 \dot{x}_2 + I_{R_4} = q_4 + C_3 \dot{x}_3, \tag{8g}$$
$$u_2 - R_3 I_{R_3} + x_2 - R_1 I_{R_1} = 0. \tag{8h}$$

From (8a) and (8c):

$$I_{R_1} = -\frac{1}{R_1} x_1 + \frac{1}{R_1} u_1, \qquad I_{R_4} = \frac{1}{R_4} q_2. \tag{9}$$

Adding (8a) and (8b):

$$u_1 - x_1 - q_1 - R_2 q_3 = 0. \tag{10}$$

From (8e), (8g) and (8c)

$$R_3 C_2 \dot{x}_2 - (R_3 + R_5) C_3 \dot{x}_3 - x_3 + \frac{R_3}{R_4} q_2 - R_3 q_4 + u_2 = 0. \tag{11}$$

From (8a) and (8f):

$$C_1 \dot{x}_1 = -\frac{1}{R_1} x_1 + \frac{1}{R_1} u_1 + q_3 - q_4 + \frac{1}{R_4} q_2 + C_2 \dot{x}_2. \tag{12}$$

From (8h) and (8g):

$$R_3 C_2 \dot{x}_2 - R_3 C_3 \dot{x}_3 + x_2 + x_1 + \frac{R_3}{R_4} q_2 - R_3 q_4 - u_1 + u_2 = 0. \tag{13}$$

Subtracting (13) from (11):

$$-R_5C_3\dot{x}_3 - x_3 - x_2 - x_1 + u_1 = 0, \tag{14}$$

so

$$\dot{x}_3 = -\frac{1}{R_5C_3}x_1 - \frac{1}{R_5C_3}x_2 - \frac{1}{R_5C_3}x_3 + \frac{1}{R_5C_3}u_1. \tag{15}$$

Substituting (15) in (11):

$$\begin{aligned}\dot{x}_2 = &-\frac{R_3+R_5}{R_3R_5C_2}x_1 - \frac{R_3+R_5}{R_3R_5C_2}x_2 + \left(\frac{1}{R_3C_2} - \frac{R_3+R_5}{R_3R_5C_2}\right)x_3 \\ &- \frac{1}{R_4C_2}q_2 + \frac{1}{C_2}q_4 + \frac{R_3+R_5}{R_3R_5C_2}u_1 - \frac{1}{R_3C_2}u_2.\end{aligned} \tag{16}$$

Equivalently,

$$\begin{aligned}\dot{x}_2 = &-\frac{1}{C_2}\left(\frac{1}{R_3}+\frac{1}{R_5}\right)x_1 - \frac{1}{C_2}\left(\frac{1}{R_3}+\frac{1}{R_5}\right)x_2 - \frac{1}{R_5C_2}x_3 \\ &- \frac{1}{R_4C_2}q_2 + \frac{1}{C_2}q_4 + \frac{1}{C_2}\left(\frac{1}{R_3}+\frac{1}{R_5}\right)u_1 - \frac{1}{R_3C_2}u_2.\end{aligned} \tag{17}$$

Substituting (17) in (12):

$$\begin{aligned}\dot{x}_1 = &-\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_3}+\frac{1}{R_5}\right)x_1 - \frac{1}{C_1}\left(\frac{1}{R_3}+\frac{1}{R_5}\right)x_2 - \frac{1}{R_5C_1}x_3 \\ &+ \frac{1}{C_1}q_3 + \frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_3}+\frac{1}{R_5}\right)u_1 - \frac{1}{R_3C_1}u_2.\end{aligned} \tag{18}$$

From (18), (17) and (15) the state matrices are extracted as:

$$\mathbf{A} = \begin{bmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_3}+\frac{1}{R_5}\right) & -\frac{1}{C_1}\left(\frac{1}{R_3}+\frac{1}{R_5}\right) & -\frac{1}{R_5C_1} \\ -\frac{1}{C_2}\left(\frac{1}{R_3}+\frac{1}{R_5}\right) & -\frac{1}{C_2}\left(\frac{1}{R_3}+\frac{1}{R_5}\right) & -\frac{1}{R_5C_2} \\ -\frac{1}{R_5C_3} & -\frac{1}{R_5C_3} & -\frac{1}{R_5C_3} \end{bmatrix}, \tag{19a}$$

$$\mathbf{B} = \begin{bmatrix} \frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_3}+\frac{1}{R_5}\right) & -\frac{1}{R_3C_1} \\ \frac{1}{C_2}\left(\frac{1}{R_3}+\frac{1}{R_5}\right) & -\frac{1}{R_3C_2} \\ \frac{1}{R_5C_3} & 0 \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & -\frac{1}{R_4C_2} & 0 & \frac{1}{C_2} \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{19b}$$

From (8d) and (10) the linear constraint matrices are extracted as:

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & R_2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{E} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{20}$$

From (6j), (6m), and (15):

$$y = R_5C_3\dot{x}_3 = -x_1 - x_2 - x_3 + u_1. \tag{21}$$

And the output matrices are extracted as:

$$\mathbf{M} = \begin{bmatrix} -1 & -1 & -1 \end{bmatrix}, \qquad \mathbf{L} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad \mathbf{N} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}. \tag{22}$$

These, alongside the nonlinear constraints (exact Ebers-Moll model for the BJT transistor) and Jacobian:

$$\boldsymbol{\eta}(\mathbf{q}) = \begin{bmatrix} q_3 - I_S\left(\left(e^{q_1/V_T} - e^{q_2/V_T}\right) + \dfrac{1}{\beta_F}\left(e^{q_1/V_T} - 1\right)\right) \\ q_4 - I_S\left(\left(e^{q_1/V_T} - e^{q_2/V_T}\right) - \dfrac{1}{\beta_R}\left(e^{q_2/V_T} - 1\right)\right) \end{bmatrix}, \tag{23a}$$

$$\frac{d\boldsymbol{\eta}(\mathbf{q})}{d\mathbf{q}} = \frac{I_S}{V_T}\begin{bmatrix} -\left(1 + \dfrac{1}{\beta_F}\right)e^{q_1/V_T} & e^{q_2/V_T} & \dfrac{V_T}{I_S} & 0 \\ -e^{q_1/V_T} & \left(1 + \dfrac{1}{\beta_R}\right)e^{q_2/V_T} & 0 & \dfrac{V_T}{I_S} \end{bmatrix}, \tag{23b}$$

define the complete model for the common-emitter amplifier circuit.

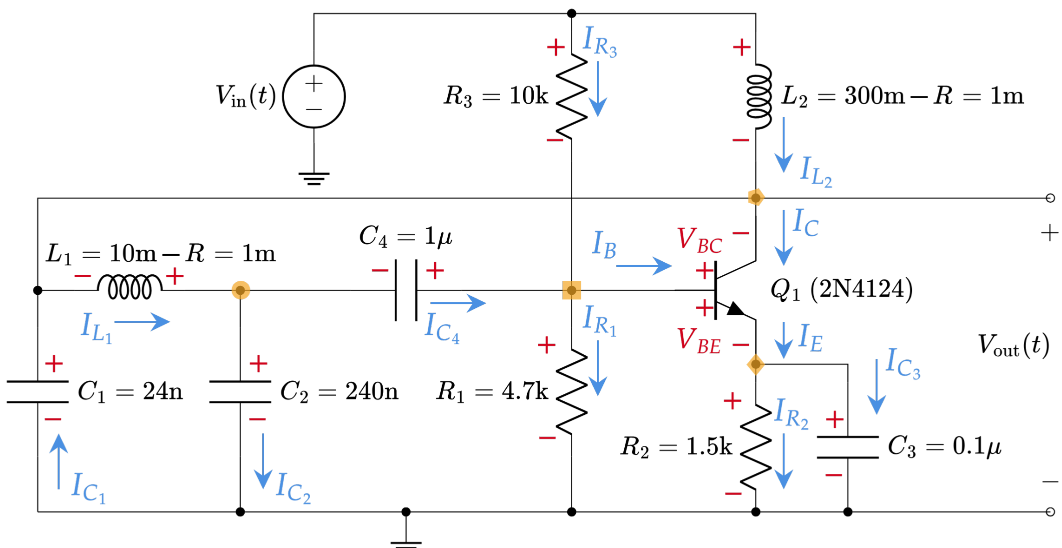


Figure 3: Colpitts oscillator circuit. Inductors are assumed to have a small in-series resistance R.

Circuit equations using the quantities defined in Figure 3:

$$V_{C_1} + V_{L_1} - I_{L_1}R - V_{C_2} = 0, \tag{24a}$$

$$V_{C_2} + V_{C_4} - I_{R_1}R_1 = 0, \tag{24b}$$

$$I_{R_1}R_1 - V_{BE} - I_{R_2}R_2 = 0, \tag{24c}$$

$$I_{R_2}R_2 = V_{C_3}, \tag{24d}$$

$$V_{\text{in}} - I_{R_3}R_3 - I_{R_1}R_1 = 0, \tag{24e}$$

$$I_{R_3}R_3 - V_{L_2} - I_{L_2}R + V_{BC} = 0, \tag{24f}$$

$$I_{L_1} = I_{C_4} + I_{C_2}, \tag{24g}$$

$$I_{C_4} + I_{R_3} = I_{R_1} + I_B, \tag{24h}$$

$$I_E = I_{R_2} + I_{C_3}, \tag{24i}$$

$$I_{C_1} + I_{L_2} = I_{L_1} + I_C, \tag{24j}$$
$$I_B = I_E - I_C, \tag{24k}$$
$$V_{\text{out}} = I_{R_4} R_4, \tag{24l}$$
$$I_{C_1} = -C_1 \dot{V}_{C_1}, \tag{24m}$$
$$I_{C_2} = C_2 \dot{V}_{C_2}, \tag{24n}$$
$$I_{C_3} = C_3 \dot{V}_{C_3}, \tag{24o}$$
$$I_{C_4} = -C_4 \dot{V}_{C_4}, \tag{24p}$$
$$V_{L_1} = -L_1 \dot{I}_{L_1}, \tag{24q}$$
$$V_{L_2} = L_2 \dot{I}_{L_2}. \tag{24r}$$

Change of variables:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} V_{C_1} \\ V_{C_2} \\ V_{C_3} \\ V_{C_4} \\ I_{L_1} \\ I_{L_2} \end{bmatrix}, \qquad u = V_{\text{in}}, \qquad y = V_{\text{out}}, \qquad \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} V_{BE} \\ V_{BC} \\ I_E \\ I_C \end{bmatrix}. \tag{25}$$

Under (25), equations (24a)–(24r) become:

$$x_1 - L_1 \dot{x}_5 - R x_5 - x_2 = 0, \tag{26a}$$
$$x_2 + x_4 - R_1 I_{R_1} = 0, \tag{26b}$$
$$R_1 I_{R_1} - q_1 - R_2 I_{R_2} = 0, \tag{26c}$$
$$R_2 I_{R_2} = x_3, \tag{26d}$$
$$u - R_3 I_{R_3} - R_1 I_{R_1} = 0, \tag{26e}$$
$$R_3 I_{R_3} - L_2 \dot{x}_6 - R x_6 + q_2 = 0, \tag{26f}$$
$$x_5 = -C_4 \dot{x}_4 + C_2 \dot{x}_2, \tag{26g}$$
$$-C_4 \dot{x}_4 + I_{R_3} = I_{R_1} + q_3 - q_4, \tag{26h}$$
$$q_3 = I_{R_2} + C_3 \dot{x}_3 = \frac{1}{R_2} x_3 + C_3 \dot{x}_3, \tag{26i}$$
$$-C_1 \dot{x}_1 + x_6 = x_5 + q_4, \tag{26j}$$
$$y = x_1. \tag{26k}$$

From (26a):

$$\dot{x}_5 = \frac{1}{L_1} x_1 - \frac{1}{L_1} x_2 - \frac{R}{L_1} x_5. \tag{27}$$

Adding (26c) and (26d):

$$R_1 I_{R_1} = q_1 + x_3, \tag{28}$$

then adding this to (26b):

$$x_2 + x_4 - x_3 - q_1 = 0. \tag{29}$$

From (28) and (26e):

$$I_{R_1} = \frac{1}{R_1}x_3 + \frac{1}{R_1}q_1, \qquad I_{R_3} = -\frac{1}{R_3}x_3 + \frac{1}{R_3}u - \frac{1}{R_3}q_1. \tag{30}$$

Adding (26e) and (26f), alongside (30):

$$u - x_3 - q_1 - L_2\dot{x}_6 - Rx_6 + q_2 = 0, \tag{31}$$

so

$$\dot{x}_6 = -\frac{1}{L_2}x_3 - \frac{R}{L_2}x_6 + \frac{1}{L_2}u - \frac{1}{L_2}q_1 + \frac{1}{L_2}q_2. \tag{32}$$

From (26d) and (26i):

$$\dot{x}_3 = -\frac{1}{R_2C_3}x_3 + \frac{1}{C_3}q_3. \tag{33}$$

From (26h) and (30):

$$\dot{x}_4 = -\left(\frac{1}{R_1C_4} + \frac{1}{R_3C_4}\right)x_3 + \frac{1}{R_3C_4}u - \left(\frac{1}{R_1C_4} + \frac{1}{R_3C_4}\right)q_1 - \frac{1}{C_4}q_3 + \frac{1}{C_4}q_4. \tag{34}$$

From (26g) and (34):

$$\dot{x}_2 = \frac{C_4}{C_2}\dot{x}_4 + \frac{1}{C_2}x_5, \tag{35}$$

so

$$\dot{x}_2 = -\left(\frac{1}{R_1C_2} + \frac{1}{R_3C_2}\right)x_3 + \frac{1}{C_2}x_5 + \frac{1}{R_3C_2}u - \left(\frac{1}{R_1C_2} + \frac{1}{R_3C_2}\right)q_1 - \frac{1}{C_2}q_3 + \frac{1}{C_2}q_4. \tag{36}$$

From (26j):

$$\dot{x}_1 = -\frac{1}{C_1}x_5 + \frac{1}{C_1}x_6 - \frac{1}{C_1}q_4. \tag{37}$$

Additionally, using Kirchhoff's voltage Law around $C_2 - C_4 - V_{BE} - C_3$ and around $C_1 - V_{BC} - C_4 - C_2$:

$$q_1 = x_2 - x_3 + x_4, \tag{38a}$$

$$q_2 = -x_1 + x_2 + x_4. \tag{38b}$$

From (37), (36), (33), (34), (27), and (32), the state matrices are extracted as:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{1}{C_1} & \frac{1}{C_1} \\ 0 & 0 & -\left(\frac{1}{R_1C_2} + \frac{1}{R_3C_2}\right) & 0 & \frac{1}{C_2} & 0 \\ 0 & 0 & -\frac{1}{R_2C_3} & 0 & 0 & 0 \\ 0 & 0 & -\left(\frac{1}{R_1C_4} + \frac{1}{R_3C_4}\right) & 0 & 0 & 0 \\ \frac{1}{L_1} & -\frac{1}{L_1} & 0 & 0 & -\frac{R}{L_1} & 0 \\ 0 & 0 & -\frac{1}{L_2} & 0 & 0 & -\frac{R}{L_2} \end{bmatrix}, \tag{39a}$$

$$
\mathbf{B} = \begin{bmatrix} 0 \\ \dfrac{1}{R_3 C_2} \\ 0 \\ \dfrac{1}{R_3 C_4} \\ 0 \\ \dfrac{1}{L_2} \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} 0 & 0 & 0 & -\dfrac{1}{C_1} \\ -\left(\dfrac{1}{R_1 C_2} + \dfrac{1}{R_3 C_2}\right) & 0 & -\dfrac{1}{C_2} & \dfrac{1}{C_2} \\ 0 & 0 & \dfrac{1}{C_3} & 0 \\ -\left(\dfrac{1}{R_1 C_4} + \dfrac{1}{R_3 C_4}\right) & 0 & -\dfrac{1}{C_4} & \dfrac{1}{C_4} \\ 0 & 0 & 0 & 0 \\ -\dfrac{1}{L_2} & \dfrac{1}{L_2} & 0 & 0 \end{bmatrix}. \tag{39b}
$$

From (38a) and (38b) the linear constraint matrices are extracted as:

$$
\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 0 & 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \qquad \mathbf{E} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{40}
$$

From (6j), the output matrices are extracted as:

$$
\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{L} = 0, \qquad \mathbf{N} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}. \tag{41}
$$

These, alongside the nonlinear constraints (23a) and Jacobian (23b), define the complete model for the Colpitts oscillator circuit.